THE LINEAR DEPENDENCE OF FUNCTIONS OF SEVERAL VARIABLES, AND COMPLETELY INTEGRABLE SYSTEMS OF HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS*

BY

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Introduction

In the case of n functions of a single variable, the vanishing of the wronskian is the most familiar criterion for their linear dependence. The wronskian also plays an important part in connection with the theory of a single ordinary homogeneous linear differential equation of the nth order, even in questions which do not concern the linear dependence of solutions of the equation. It is the purpose of the present paper to generalize the fundamental facts connected with these topics, to the case of functions of several variables.

The characteristic property of an ordinary homogeneous linear differential equation is that any solution is expressible linearly, with constant coefficients, in terms of a fundamental set of solutions. The natural generalization to several independent variables is afforded by the completely integrable system of homogeneous linear partial differential equations, in one dependent variable, any solution of which is likewise linearly dependent on a finite number of solutions of the system. It is such systems which are discussed in the latter part of this paper; the subject of the linear dependence of functions of several variables is developed in the first part, chiefly with a view to its subsequent application to completely integrable systems.

In our discussion of linear dependence, we consider throughout n functions $y_1, y_2, \cdots, y_n$ of p independent variables $u_1, u_2, \cdots, u_p$. Functions and variables may be either real or complex; if any of the variables be complex, we suppose the functions to be analytic in those variables. All of the main theorems, however, are stated and proved under the supposition that the independent variables are real; the modifications—usually simplifications and omissions—are easily supplied for cases in which one or more of the independent variables is complex. The n functions, then, of the real variables

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$u_1, u_2, \cdots, u_p$, exist, together with such of their derivatives as come into the discussion, in a p-dimensional region, open or closed, which for simplicity may be taken to be a p-dimensional parallelopiped. This region we denote throughout by A. We shall suppose that all partial differentiations taken with respect to more than one of the independent variables are independent of the order in which the differentiations are effected. Thus, by the existence of $\partial^2 y/\partial u \partial v$ we imply the existence of $\partial^2 y/\partial v \partial u$ and the identity of these two derivatives.

In the case of functions of several variables, there is no one determinant which may properly be taken as the generalization of the wronskian. In fact, since there is no relation between the number of functions and the number of independent variables, no definite form can be prescribed for such a determinant; and moreover, the difficulty in this direction is increased by the fact that we do not assume our functions to be analytic. However, in the general criteria which we obtain concerning the linear dependence of n functions, a matrix takes the place of the wronskian. This matrix contains n columns, and each row is formed of the same derivative of the n functions. Evidently, the identical vanishing of all the n-rowed determinants of such a matrix is a necessary condition for the linear dependence of the n functions. The sufficient conditions at which we arrive are expressed in terms of the vanishing of the n-rowed determinants of a matrix of the kind just described, and are therefore also necessary—a point which we shall not mention in the body of the paper. The case of n functions of a single variable is of course included in our general theory; the matrix for such functions may have n or more rows, and if it has only n the determinant of the matrix is precisely the wronskian of the n functions.

So far as we are aware, the only completely integrable systems which have been studied are of two kinds. Of these, the only non-analytic ones are of the first order, and are in fact simply systems of total differential equations.* For systems consisting of equations of order higher than the first, an analytic character has been presupposed† in establishing the existence of solutions. By a mere change of notation, however, we cause such a system to take on the form of a system of total differential equations, so that the existence of solutions may be proved in the non-analytic case for completely integrable systems consisting of equations of unrestricted orders.

Although for a set of functions of several variables there is in general no distinctive determinant, formed from partial derivatives of the functions,

* Cf., for instance, Mayer, *Ueber unbeschränkt integrable Systeme von linearen totalen Differentialgleichungen und die simultane Integration linearer partieller Differentialgleichungen*, Mathematische Annalen, vol. 5 (1872), pp. 448–470.

† Cf. C. Riquier, *Les systèmes d'équations aux dérivées partielles*, Paris, Gauthier-Villars, 1910.

which might play the rôle of a wronskian, we have been able to define a wronskian for a fundamental set of solutions of a completely integrable system. By means of this wronskian, if we may so term it, we find certain properties of homogeneous linear completely integrable systems quite analogous to those of an ordinary differential equation, linear and homogeneous, of the nth order. An important result in this connection is a proof that a large and important class of completely integrable systems of the kind considered may be put into a so-called canonical form by a certain transformation of the dependent variable. The theorem is analogous to the familiar one in the case of an ordinary differential equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = 0,$$

according to which the equation can be reduced in essentially but one way to a unique form for which the coefficient p_1 is zero, by a transformation of the dependent variable alone of the form $\bar{y} = \lambda(x) y$.

The reduction, just referred to, of a completely integrable system to a canonical form, is very useful in certain geometric questions. Completely integrable systems have been important in differential geometry, but not until recently has systematic use been made of them in connection with that branch of the subject for which they are most peculiarly fitted, viz., projective differential geometry. We owe to Wilczynski a very general method for dealing with questions in that field; in fact, according to this method, the projective differential geometry of a configuration is equivalent to the theory of the invariants and covariants, under certain continuous groups, of a completely integrable system, the configuration being defined by a fundamental set of solutions of the system. The continuous groups which give rise to the invariants and covariants are generally broken up into two parts: a transformation of the independent variables, and a transformation of the dependent variable. The coefficients of the canonical form are invariant under this latter transformation, and in terms of these coefficients and their derivatives may be expressed all quantities having the same invariantive property. It is therefore highly desirable to have a method for computing these seminvariants, as they are called; we show that *this can be done in a purely mechanical way for certain completely integrable systems.* In the various particular problems studied by Wilczynski and others, this method is the one which has always been followed.

As an illustration, we show in § 7 how the foregoing may be applied to the theory of curvilinear coördinates in space of n dimensions.

1. The fundamental theorem

Before considering the linear dependence of n functions, it will be convenient to prove the following theorem for two functions:

Theorem I. *Let y_1 and y_2 be functions of the p independent variables $u_1, u_2, \cdots, u_p$ for which all partial derivatives of the first order, $\partial y_1/\partial u_k$, $\partial y_2/\partial u_k$, $(k = 1, 2, \cdots, p)$ exist throughout the region A. Suppose, further, that one of the functions, say y_1, vanishes at no point of A. Then if all the two-rowed determinants in the matrix*

$$\left\| \begin{matrix} y_1, & y_2 \\ \frac{\partial y_1}{\partial u_1}, & \frac{\partial y_2}{\partial u_1} \\ \frac{\partial y_1}{\partial u_2}, & \frac{\partial y_2}{\partial u_2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{\partial y_1}{\partial u_p}, & \frac{\partial y_2}{\partial u_p} \end{matrix} \right\|$$

vanish identically in A, y_1 and y_2 are linearly dependent in A, and in fact

$$y_2 \equiv c y_1.$$

For, if

$$y_1 \frac{\partial y_2}{\partial u_k} - y_2 \frac{\partial y_1}{\partial u_k} \equiv 0 \qquad (k = 1, 2, \cdots, p),$$

we have on dividing by y_1^2,

$$\frac{\partial}{\partial u_k}\left(\frac{y_2}{y_1}\right) \equiv 0 \qquad (k = 1, 2, \cdots, p),$$

whence

$$\frac{y_2}{y_1} \equiv c, \qquad y_2 = c y_1,$$

where c is a constant.

We might of course have made the statement of the hypothesis somewhat weaker, at the sacrifice of smoothness of statement, since all the two-rowed determinants of the matrix vanish if all those vanish in which the first row of the matrix constitutes a row.

We note at once that in the case of functions of more than one variable there does not exist a single determinant which may properly be called the wronskian. We might of course speak of the two-rowed determinants of the above matrix as wronskians, but in the case of more than two dependent variables, which we shall now consider, new difficulties arise. First of all, there is no connection between the number of functions and the number of independent variables. Moreover, the various partial derivatives of the functions may exist in a very unsymmetric way, that is, a function may for instance be analytic in some of the variables, and may have only a few derivatives in the others. Consequently, there is not even a set of determinants, the structure of which we may describe once for all, which might take the place of the single wronskian in the case of functions of one variable. As a matter of fact, we shall see presently that very useful sufficient conditions for linear dependence may be stated, in which the partial derivatives with respect to some of the variables need exist only to the first order, while those with respect to other variables may be of higher orders.

We proceed to the fundamental theorem, of which Theorem I is a special case. Let $y_1, y_2, \cdots, y_n$ be functions of the p independent variables $u_1, u_2, \cdots, u_p$. Let us denote by $y_i^{(1)}$, $y_i^{(2)}$, etc., partial derivatives of y_i, of any order or kind whatever. As a matter of fact, it will not be necessary to specify just what derivative of y_i is denoted by the symbol $y_i^{(j)}$. If each one of the set of functions y_i $(i = 1, 2, \cdots, n)$ possesses a certain partial derivative, we shall say that *the set of functions possesses that derivative.* Thus, if $\partial^2 y_i/\partial u_1 \partial u_2$, $(i = 1, 2, \cdots, n)$ exist, the set of functions possesses the derivative $\partial^2 y/\partial u_1 \partial u_2$.

We shall frequently have occasion to consider matrices of the form

$$\left\| \begin{matrix} y_1, & y_2, & \cdots, & y_r \\ y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_r^{(1)} \\ y_1^{(2)}, & y_2^{(2)}, & \cdots, & y_r^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(s)}, & y_2^{(s)}, & \cdots, & y_r^{(s)} \end{matrix} \right\|,$$

in which *the first row consists of the functions* $y_1, y_2, \cdots, y_r$, and the other s rows of derivatives of these functions. We shall denote throughout the matrix just written by the symbol

$$M_s(y_1, y_2, \cdots, y_r).$$

We may now state the fundamental theorem:

THEOREM II. *Let the set of n functions $y_1, y_2, \cdots, y_n$ of the p independent variables $u_1, u_2, \cdots, u_p$ possess enough partial derivatives, of any orders whatever, to form a matrix*

$$M \equiv M_{n-2}(y_1, y_2, \cdots, y_n),$$

of n columns and $n - 1$ rows, in which at least one of the $(n - 1)$-rowed determinants, say

$$W_n \equiv \begin{vmatrix} y_1, & y_2, & \cdots, & y_{n-1} \\ y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_{n-1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \cdots, & y_{n-1}^{(n-2)} \end{vmatrix},$$

vanishes nowhere in A. Suppose, further, that all of the first derivatives of each of the elements of the above matrix M exist, and adjoin to the matrix M such of these derivatives as do not already appear in M, to form the new matrix

$$M' \equiv M_q(y_1, y_2, \cdots, y_n),$$

which has n columns and at least n rows, so that $q \geqq n-1$. Then if all the n-rowed determinants of the matrix M' in which the determinant W_n is a first minor vanish identically in A, the functions $y_1, y_2, \cdots, y_n$ are linearly dependent in A, and in fact

$$y_n \equiv c_1 y_1 + c_2 y_2 + \cdots + c_{n-1} y_{n-1},$$

the c's being constants.

Let us denote by $W_1, W_2, \cdots, W_n$ the $(n-1)$-rowed determinants of the matrix M, in such wise that W_i is the determinant which does not contain the ith column of M. If we write $y_i^{(0)} = y_i$, we have by the hypothesis

$$W_1 y_1^{(j)} + W_2 y_2^{(j)} + \cdots + W_n y_n^{(j)} \equiv 0 \quad (j = 0, 1, \cdots, q). \tag{1}$$

Differentiating each of the first $n-1$ of these equations with respect to u_k, we obtain the equations

$$\begin{aligned} &\frac{\partial W_1}{\partial u_k} y_1^{(j)} + \frac{\partial W_2}{\partial u_k} y_2^{(j)} + \cdots + \frac{\partial W_n}{\partial u_k} y_n^{(j)} \\ &\qquad + W_1 \frac{\partial y_1^{(j)}}{\partial u_k} + W_2 \frac{\partial y_2^{(j)}}{\partial u_k} + \cdots + W_n \frac{\partial y_n^{(j)}}{\partial u_k} \equiv 0 \\ &\qquad\qquad (j = 0, 1, \cdots, n-2). \end{aligned} \tag{2}$$

But since all of the first derivatives of the elements of the matrix M occur in the matrix M', the last part of each of the relations (2) vanishes because of (1), so that

$$\frac{\partial W_1}{\partial u_k} y_1^{(j)} + \frac{\partial W_2}{\partial u_k} y_2^{(j)} + \cdots + \frac{\partial W_n}{\partial u_k} y_n^{(j)} \equiv 0 \quad (j = 0, 1, \cdots, n-2).$$

Multiplying the first of these by the first minor of $y_1^{(0)}$ $(\equiv y_1)$ in W_n, the second by the first minor of $y_1^{(1)}$ in W_n, etc., the last by the first minor of $y_1^{(n-2)}$ in W_n, and adding the results, we find that

$$\frac{\partial W_1}{\partial u_k} W_n - \frac{\partial W_n}{\partial u_k} W_1 \equiv 0.$$

But this holds for every k; hence by Theorem I, since $W_n \neq 0$ anywhere in A, we have

$$W_1 = -c_1 W_n.$$

In like manner,

$$W_2 = -c_2 W_n,$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$W_{n-1} = -c_{n-1} W_n,$$

the c's being constants. Consequently, from the identity

$$W_1 y_1 + W_2 y_2 + \cdots + W_n y_n \equiv 0,$$

we have

$$W_n(-c_1 y_1 - c_2 y_2 - \cdots - c_{n-1} y_{n-1} + y_n) \equiv 0.$$

Since W_n is nowhere zero in A, it follows that

$$y_n \equiv c_1 y_1 + c_2 y_2 + \cdots + c_{n-1} y_{n-1},$$

whence our theorem.*

It should be noted that if, as stated in the hypothesis, all the n-rowed determinants of M' of which the determinant W_n is a first minor vanish, then *all the n-rowed determinants of M' vanish.* This follows from the non-vanishing of W_n.†

As an example of the application of Theorem II, consider the case of four functions of two independent variables, say

$$y_i = f_i(u, v) \qquad (i = 1, 2, 3, 4).$$

Then if in the matrix

$$\left\| \begin{array}{cccc} y_1, & y_2, & y_3, & y_4 \\ \dfrac{\partial y_1}{\partial u}, & \dfrac{\partial y_2}{\partial u}, & \dfrac{\partial y_3}{\partial u}, & \dfrac{\partial y_4}{\partial u} \\ \dfrac{\partial y_1}{\partial v}, & \dfrac{\partial y_2}{\partial v}, & \dfrac{\partial y_3}{\partial v}, & \dfrac{\partial y_4}{\partial v} \end{array} \right\|$$

at least one of the three-rowed determinants vanishes nowhere in the region A, a necessary and sufficient condition for the linear dependence of y_1, y_2, y_3, y_4 is the identical vanishing in A of all the four-rowed determinants of

* This proof is a direct generalization of the corresponding one given by Frobenius for functions of a single variable, in the case of the wronskian. Cf. Bôcher, *Certain cases in which the vanishing of the wronskian is a sufficient condition for linear dependence*, these Transactions, vol. 2 (1901), pp. 139–149, § 1.

† Cf. Bôcher, *Introduction to Higher Algebra*, p. 54, for the general theorem on matrices involved here.

the matrix formed by adjoining to the above one all of the derivatives of the second order of y_1, y_2, y_3, y_4. If, however, at least one of the three-rowed determinants in the matrix

$$\begin{Vmatrix} y_1, & y_2, & y_3, & y_4 \\ \frac{\partial y_1}{\partial u}, & \frac{\partial y_2}{\partial u}, & \frac{\partial y_3}{\partial u}, & \frac{\partial y_4}{\partial u} \\ \frac{\partial^2 y_1}{\partial u^2}, & \frac{\partial^2 y_2}{\partial u^2}, & \frac{\partial^2 y_3}{\partial u^2}, & \frac{\partial^2 y_4}{\partial u^2} \end{Vmatrix}$$

vanishes nowhere in A, then a necessary and sufficient condition for the linear dependence of y_1, y_2, y_3, y_4 is the identical vanishing of all the four-rowed determinants of the matrix

$$\begin{Vmatrix} y_1, & \frac{\partial y_1}{\partial u}, & \frac{\partial y_1}{\partial v}, & \frac{\partial^2 y_1}{\partial u^2}, & \frac{\partial^2 y_1}{\partial u \partial v}, & \frac{\partial^3 y_1}{\partial u^3}, & \frac{\partial^3 y_1}{\partial u^2 \partial v} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_4, & \frac{\partial y_4}{\partial u}, & \frac{\partial y_4}{\partial v}, & \frac{\partial^2 y_4}{\partial u^2}, & \frac{\partial^2 y_4}{\partial u \partial v}, & \frac{\partial^3 y_4}{\partial u^3}, & \frac{\partial^3 y_4}{\partial u^2 \partial v} \end{Vmatrix},$$

so that the existence of even $\partial^2 y_i / \partial v^2$ is not presupposed.

In the fundamental theorem concerning the linear dependence of n functions of a single variable, the identical vanishing of the wronskian is shown to be sufficient for linear dependence provided the wronskian of $n - 1$ of the functions is not zero anywhere in the interval of the independent variable under consideration. This evidently comes as a special case under Theorem II. But Theorem II also affords other sufficient criteria in the case of a single independent variable. An example will show the general nature of these criteria. Let y, z, w be functions of the real variable x in an interval I. Denoting by accents differentiation with respect to x, let us suppose that the determinant

$$\begin{vmatrix} y, & z \\ y''', & z''' \end{vmatrix}$$

vanishes nowhere in I. Then a sufficient condition for the linear dependence of y, z, w is the identical vanishing in I of the two determinants

$$\begin{vmatrix} y, & z, & w \\ y', & z', & w' \\ y''', & z''', & w''' \end{vmatrix}, \qquad \begin{vmatrix} y, & z, & w \\ y''', & z''', & w''' \\ y^{\text{IV}}, & z^{\text{IV}}, & w^{\text{IV}} \end{vmatrix}.$$

However, in any case such as the present one, a very general theorem concerning linear dependence, due to D. R. Curtiss, is applicable.*

Before leaving the fundamental theorem, it may be well to consider the case of analytic functions. For n analytic functions of a single variable, the identical vanishing of the wronskian is a sufficient condition for linear dependence, and nothing need be demanded as to the non-vanishing of the wronskian of $n - 1$ of the functions. This follows from the fact that for analytic functions linear dependence in any subregion implies linear dependence throughout their common region of definition. If in Theorem II the functions involved are analytic, an obvious weakening of the hypotheses may be made; for instance, it is sufficient to know that *all the n-rowed determinants of the matrix M' which contain the elements of W_n vanish throughout an n-dimensional sub-region of A which contains a single point at which W_n is different from zero.* Such would be the most general theorem that may be stated in this case; however, by strengthening the hypotheses sufficiently we may obtain a theorem in which no demand is made concerning the non-vanishing of W_n. A very elegant theorem of this kind is the following, which is due to O. D. Kellogg:†

Let $y_1, y_2, \cdots, y_n$ *be analytic functions of the p independent variables* $u_1, u_2, \cdots, u_p$*—real or complex—having a common region of definition* A, *and let* $M_q(y_1, y_2, \cdots, y_n)$ *be the matrix formed from the functions* $y_1, y_2, \cdots, y_n$ *and all their partial derivatives up to and including those of order* $n - 1$. *Then the functions* $y_1, y_2, \cdots, y_n$ *are linearly dependent in* A *if and only if all of the n-rowed determinants of the matrix* M_q *vanish throughout* A.

2. Two theorems concerning the linear dependence of n functions

In the present section we shall prove two theorems, which are generalizations of theorems given by Bôcher‡ for functions of a single variable. They will be useful later, in connection with completely integrable systems of partial differential equations, just as Bôcher's theorems are in the case of ordinary linear differential equations, of which completely integrable systems are a generalization to several independent variables.

Theorem III. *Let* $y_1, y_2, \cdots, y_n$ *be n functions of the variables* $u_1, u_2, \cdots, u_p$. *Suppose that the set of functions possesses the* $n - 1$ *partial derivatives* $y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(n-1)}$, $(i = 1, 2, \cdots, n)$, *so that the function*

* Cf. Theorem VI, p. 292, of his article, *The vanishing of the wronskian and the problem of linear dependence,* Mathematische Annalen, vol. 65 (1908), pp. 282–298.

† *Nomograms with points in alignment,* Zeitschrift für Mathematik und Physik, vol. 63 (1914), pp. 160–161. The statement there given is equivalent to ours, if the proper interpretation is made of Prof. Kellogg's definition of the rank of a matrix.

‡ Cf. Theorems V and VI of the paper already cited, these Transactions, vol. 2 (1901), pp. 139–149.

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where the c's are constants, has the same derivatives, $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$. *Suppose also that no function of the form of* y, *other than zero, vanishes together with its derivatives* $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ *at any point of* A. *Then if the determinant*

$$W \equiv \begin{vmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{vmatrix}$$

vanishes at any point $(a_1, a_2, \cdots, a_p)$ *of* A, *the functions* $y_1, y_2, \cdots, y_n$ *are linearly dependent.*

For, since $W(a_1, a_2, \cdots, a_p) = 0$, there exist n constants $c_1, c_2, \cdots, c_n$ not all zero, such that

$$c_1 y_1^{(j)}(a_1, \cdots, a_p) + c_2 y_2^{(j)}(a_1, \cdots, a_p) + \cdots + c_n y_n^{(j)}(a_1, \cdots, a_p) = 0 \qquad (j = 0, 1, \cdots, n-1).$$

so that the function $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ vanishes together with its derivatives $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ at a point of A, and must therefore be identically zero.

THEOREM IV. *Let* $y_1, y_2, \cdots, y_n$ *be a set of n functions of the p variables* $u_1, u_2, \cdots, u_p$ *which throughout A possesses r partial derivatives* $(r > n - 1)$,

$$y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(r)} \qquad (i = 1, 2, \cdots, n).$$

Suppose that from among these derivatives may be selected at least $n - 2$, *say* $y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(m)}$ $(m \geqq n - 2)$, *such that all of their first derivatives occur among the set* $y_i^{(1)}, y_i^{(2)}, \cdots, y_i^{(r)}$, *and are bounded in* A. *Let*

$$M \equiv M_m(y_1, y_2, \cdots, y_n) \qquad (m \geqq n - 2),$$

and suppose that

$$M' \equiv M_q(y_1, y_2, \cdots, y_n) \qquad (q > m)$$

is the matrix formed from M by adjoining to the elements of M those of their first derivatives which do not already appear in M. Assume also that if for any ν *columns of M* $(\nu \leqq n)$ *all the* ν*-rowed determinants containing elements of the first row of M vanish identically, then all the* ν*-rowed determinants formed from the same* ν *columns of* M', *and containing elements of the first row of M, and of* $\nu - 2$ *other rows of M, vanish identically.** *Suppose, further, that no function, other than zero, of the form*

* Some remarks concerning this part of the hypothesis will be found after the proof of the theorem.

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where the c's are constants, vanishes together with its derivatives $y^{(1)}, y^{(2)}, \cdots, y^{(r)}$ *at any point of* A. *Then the identical vanishing of all those n-rowed determinants of* M' *in which appear elements of the first row and* $n - 2$ *other rows of* M *is a sufficient condition that the functions* $y_1, y_2, \cdots, y_n$ *be linearly dependent.*

Suppose first that one of the $(n - 1)$-rowed determinants of the matrix M containing elements of the first row of M does not vanish identically. We may without loss of generality take for this determinant the one which appears in the upper left-hand corner of M. As in § 1, we denote this determinant by W_n. Then at some point $(a_1, a_2, \cdots, a_p)$ of A, W_n is different from zero. But W_n is continuous in A, because all its partial derivatives of the first order exist and are bounded in A*; consequently, it differs from zero in some neighborhood of the point $(a_1, a_2, \cdots, a_p)$. By Theorem II, then, the functions $y_1, y_2, \cdots, y_n$ are linearly dependent in this neighborhood, i. e., there exist constants $c_1, c_2, \cdots, c_n$, not all zero, such that the function

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is zero in the said neighborhood of $(a_1, a_2, \cdots, a_p)$. But then this function vanishes at $(a_1, a_2, \cdots, a_p)$ together with its derivatives $y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, and therefore, by hypothesis, vanishes identically. Consequently the functions $y_1, y_2, \cdots, y_n$ are linearly dependent in A.

We now assume the other alternative, viz., that all of the $(n - 1)$-rowed determinants of M which contain elements of the first row of M vanish identically; in particular, those formed from the first $n - 1$ columns of M do. We recall that, by hypothesis, if all the ν-rowed determinants formed from the first ν columns of M and containing elements of the first row of M vanish identically, then the determinants formed from the first ν columns of M', and containing elements of the first row and of $\nu - 2$ other rows of M, vanish identically. We consider then in succession the first $n - 1$ columns of M, the first $n - 2$ columns, the first $n - 3$ columns, etc. Suppose that for the first $n - i$ columns of M all of the $(n - i)$-rowed determinants which contain elements of the first row of M vanish identically. We know that this happens for $i = 0$ and $i = 1$. If it happens for $i = n - 1$, we shall have $y_1 \equiv 0$, and then $y_1, y_2, \cdots, y_n$ would be linearly dependent. Suppose then that $i < n - 1$. Then there must exist a particular i, $1 \leqq i \leqq n - 2$, such that all the $(n - i)$-rowed determinants formed from the first $n - i$ columns of M and containing elements of the first row of M vanish identically, whereas

* If for a function of p variables all the partial derivatives of the first order exist and are bounded in the p-dimensional region A, the function is continuous in A with respect to all the variables taken together. Cf. Osgood, *Lehrbuch der Funktionentheorie*, vol. I, second edition (1912), page 56.

the $(n - i - 1)$-rowed determinants formed from the first $n - i - 1$ columns of M and containing elements of the first row of M do not all vanish identically. But the identical vanishing of all the $(n - i)$-rowed determinants of the kind considered, formed from the first $n - i$ columns of M, carries with it, by hypothesis, the identical vanishing of all the $(n - i)$-rowed determinants formed from the first $n - i$ columns of M' and containing elements of the first row and of $n - i - 2$ other rows of M. We may therefore apply the special case of the present theorem which we have already proved, to deduce the linear dependence of $y_1, y_2, \cdots, y_{n-i}$. Consequently, the functions $y_1, y_2, \cdots, y_n$ are also linearly dependent, and our theorem is proved.

It is to be noted that in the hypothesis of Theorem IV, a restriction has been placed upon the nature of the matrices M and M' which we did not find it necessary to impose in the previous general theorems. As may readily be seen by an example, it is only in special cases that the identical vanishing of all the ν-rowed determinants formed from ν columns of the matrix M and containing elements of the first row of M implies the identical vanishing of all the ν-rowed determinants formed from the corresponding ν columns of M' and containing elements of the first row and of $\nu - 2$ additional rows of M. Probably the most important cases of this type arise when the functions $y_1, y_2, \cdots, y_n$ are solutions of the same completely integrable system of partial differential equations. We shall have occasion, in a subsequent section, to apply Theorem IV in just such a case.

3. Completely integrable systems of homogeneous linear partial differential equations

A system of partial differential equations in one dependent variable is said to be *completely integrable* if there exists one and only one integral of the system satisfying arbitrarily chosen initial conditions involving a finite number of arbitrary constants and arbitrary functions. The completely integrable systems which we shall consider are such that a general solution is expressible linearly, with constant coefficients, in terms of a fundamental system of solutions. Thus, these completely integrable systems, which we shall suppose to contain a single dependent variable and any number of independent variables, form a natural generalization of the ordinary linear homogeneous differential equation.

So far as we are aware, the only completely integrable systems containing partial differential equations of order higher than the first which have hitherto been studied have analytic coefficients and solutions.* From the function-theoretic point of view these are of course the most important completely

* Cf., for example, C. Riquier, *Les systèmes d'équations aux dérivées partielles*, Paris, Gauthier-Villars, 1910.

integrable systems. They have found a place not only in analysis, but in geometry as well. The Gauss-Codazzi equations which form the basis for the metric differential geometry of surfaces constitute a completely integrable system, and in a later section we shall see how important are such systems in projective differential geometry.

The simplicity of the analytic case, if we disregard questions of convergence, is apparent. Let us consider, as always, a completely integrable system of linear homogeneous partial differential equations containing a single dependent variable; then by successive differentiation of the given equations, all the derivatives of the dependent variable are expressible uniquely as linear combinations of a finite number n among them, so that by well-known methods it is seen that the general solution contains exactly the same number n of arbitrary constants, which enter linearly.

We shall confine our attention throughout to linear homogeneous completely integrable systems in one dependent variable, though most of what we shall have to say may readily be extended to the case of several dependent variables. We shall not, however, suppose the coefficients in our equations to be analytic.*

Let us first construct a completely integrable system which shall have as a fundamental system of solutions an assigned set of functions. Suppose that the n functions $y_1, y_2, \cdots, y_n$ of the p independent variables $u_1, u_2, \cdots, u_p$ possess enough partial derivatives, of any kind or order whatever, to form a determinant of the nth order,

$$(3) \qquad W \equiv \begin{vmatrix} y_1, & y_2, & & \cdots, & y_n \\ y_1^{(1)}, & y_2^{(1)}, & & \cdots, & y_n^{(1)} \\ y_1^{(2)}, & y_2^{(2)}, & & \cdots, & y_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & & \cdots, & y_n^{(n-1)} \end{vmatrix},$$

which vanishes nowhere in a region A, and in which, as heretofore, we have denoted derivatives of a function y by $y^{(1)}$, $y^{(2)}$, etc., the indices being no indication of the kind or order of the derivative. Suppose also that all of the derivatives of the first order exist for all of the elements in the determinant

* Completely integrable systems in which the coefficients are not restricted to being analytic were curiously enough the first to be studied, but only such systems as consist of equations of the first order. These systems are really systems of total differential equations, and the classic work of Mayer in connection therewith is well known. See his memoir, *Ueber unbeschränkt integrable Systeme von linearen totalen Differentialgleichungen und die simultane Integration linearer partieller Differentialgleichungen,* Mathematische Annalen, vol. 5 (1872), pp. 448–470.

W. Then we may set up as follows a system of partial differential equations, of which each of the functions $y_1, y_2, \cdots, y_n$ is a solution, and such that any solution whatever, say $\bar{y}$, is of the form

$$\bar{y} \equiv c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where the c's are constants. Let us write down the system of np equations

$$
(4)\qquad
\begin{aligned}
\frac{\partial y}{\partial u_k} &= a_0^{(0,k)} y + a_1^{(0,k)} y^{(1)} + a_2^{(0,k)} y^{(2)} + \cdots + a_{n-1}^{(0,k)} y^{(n-1)},\\
\frac{\partial y^{(1)}}{\partial u_k} &= a_0^{(1,k)} y + a_1^{(1,k)} y^{(1)} + a_2^{(1,k)} y^{(2)} + \cdots + a_{n-1}^{(1,k)} y^{(n-1)},\\
\frac{\partial y^{(2)}}{\partial u_k} &= a_0^{(2,k)} y + a_1^{(2,k)} y^{(1)} + a_2^{(2,k)} y^{(2)} + \cdots + a_{n-1}^{(2,k)} y^{(n-1)},\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
\frac{\partial y^{(n-1)}}{\partial u_k} &= a_0^{(n-1,k)} y + a_1^{(n-1,k)} y^{(1)} + a_2^{(n-1,k)} y^{(2)} + \cdots + a_{n-1}^{(n-1,k)} y^{(n-1)},\\
&\qquad (k = 1, 2, \cdots, p).
\end{aligned}
$$

We may so determine the coefficients $a_i^{(j,k)}$ of this system that the functions $y_1, y_2, \cdots, y_n$ will be solutions of the system. If in any one of the equations these functions be substituted successively, n linear equations are obtained which may be solved for the coefficients a of the first mentioned equation since the determinant W is different from zero. If the coefficients a are determined in this way, the system of differential equations will surely have as solutions each of the functions $y_1, y_2, \cdots, y_n$. In fact, the system of np differential equations may be written in the compact form

$$
(5)\qquad
\begin{vmatrix}
y_1, & y_2, & \cdots, & y_n, & y\\
y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_n^{(1)}, & y^{(1)}\\
y_1^{(2)}, & y_2^{(2)}, & \cdots, & y_n^{(2)}, & y^{(2)}\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)}, & y^{(n-1)}\\
\frac{\partial y_1^{(j)}}{\partial u_k}, & \frac{\partial y_2^{(j)}}{\partial u_k}, & \cdots, & \frac{\partial y_n^{(j)}}{\partial u_k}, & \frac{\partial y^{(j)}}{\partial u_k}
\end{vmatrix} = 0
$$

$$(j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p),$$

which makes the statement evident. Moreover, it is also evident that any expression of the form

$$\bar{y} = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

will also satisfy the system. Lastly, any solution of the system must be of this form. For, let $\bar{y}$ be any solution of the system, and consider the two matrices

$$(6)\qquad \begin{aligned} M &\equiv M_{n-1}(y_1, y_2, \cdots, y_n, y), \\ M' &\equiv M_q(y_1, y_2, \cdots, y_n, y), \end{aligned}$$

where M' differs from M by the additional rows consisting of all of the partial derivatives of the first order of the elements of M which do not already appear in M. The derivatives appearing in these additional rows are precisely those which appear in the left-hand members of the system of differential equations (4), so that each of the additional rows is a linear combination of the n rows of the matrix M. Consequently, all of the $(n + 1)$-rowed determinants of M' which contain the elements of M vanish identically. But one of the n-rowed determinants of M, namely W, vanishes nowhere in A, so that by Theorem II the $n + 1$ functions $y_1, y_2, \cdots, y_n, \bar{y}$ are linearly dependent, and in fact

$$\bar{y} = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

We may state our result as follows:

THEOREM V. *If the system of partial differential equations* (4) *has n solutions* $y_1, y_2, \cdots, y_n$ *for which the determinant* (3) *vanishes nowhere in* A, *then any solution of the system will be linearly dependent on these n solutions, and any function linearly dependent on these n solutions will also be a solution.*

We have therefore obtained a completely integrable system of which the given functions $y_1, y_2, \cdots, y_n$ constitute a fundamental system of solutions. The form in which we have chosen to write the system of differential equations is not generally the simplest for any given particular case; in fact, in many cases most of the equations will be found to be merely identities, and others of them obtainable from the rest by mere differentiation. The completely integrable system will in such a case reduce to fewer than np equations. An example will make the point clearer. Suppose that the four functions

$$y_k = y_k(u_1, u_2) \qquad (k = 1, 2, 3, 4)$$

are such that the determinant

$$W = \begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ \dfrac{\partial y_1}{\partial u_1}, & \dfrac{\partial y_2}{\partial u_1}, & \dfrac{\partial y_3}{\partial u_1}, & \dfrac{\partial y_4}{\partial u_1} \\ \dfrac{\partial y_1}{\partial u_2}, & \dfrac{\partial y_2}{\partial u_2}, & \dfrac{\partial y_3}{\partial u_2}, & \dfrac{\partial y_4}{\partial u_2} \\ \dfrac{\partial^2 y_1}{\partial u_1 \partial u_2}, & \dfrac{\partial^2 y_2}{\partial u_1 \partial u_2}, & \dfrac{\partial^2 y_3}{\partial u_1 \partial u_2}, & \dfrac{\partial^2 y_4}{\partial u_1 \partial u_2} \end{vmatrix}$$

is zero at no point of the region A. Then the completely integrable system of which the four functions y_1, y_2, y_3, y_4 form a fundamental system of solutions will be

$$\frac{\partial y}{\partial u_1} = \frac{\partial y}{\partial u_1}, \qquad \frac{\partial y}{\partial u_2} = \frac{\partial y}{\partial u_2}, \qquad \frac{\partial^2 y}{\partial u_1 \partial u_2} = \frac{\partial^2 y}{\partial u_1 \partial u_2},$$

$$\frac{\partial^2 y}{\partial u_1^2} = ay + b\frac{\partial y}{\partial u_1} + c\frac{\partial y}{\partial u_2} + d\frac{\partial^2 y}{\partial u_1 \partial u_2},$$

$$\frac{\partial^2 y}{\partial u_2^2} = a' y + b'\frac{\partial y}{\partial u_1} + c'\frac{\partial y}{\partial u_2} + d'\frac{\partial^2 y}{\partial u_1 \partial u_2},$$

$$\frac{\partial^3 y}{\partial u_1^2 \partial u_2} = a'' y + b''\frac{\partial y}{\partial u_1} + c''\frac{\partial y}{\partial u_2} + d''\frac{\partial^2 y}{\partial u_1 \partial u_2},$$

$$\frac{\partial^3 y}{\partial u_1 \partial u_2^2} = a''' y + b'''\frac{\partial y}{\partial u_1} + c'''\frac{\partial y}{\partial u_2} + d'''\frac{\partial^2 y}{\partial u_1 \partial u_2};$$

we have of course presupposed the existence of all the derivatives which we have written down. The first three of the above equations are identities, and as a matter of fact the third of them ought to occur twice. The completely integrable system therefore consists essentially of four instead of $2 \times 4 = 8$ equations. Moreover, if $dd' - 1$ is nowhere zero in A, the last two of the differential equations may be obtained from the two immediately preceding them by differentiation and algebraic processes. The completely integrable system of eight equations in this case reduces therefore to the two equations

$$\frac{\partial^2 y}{\partial u_1^2} = ay + b\frac{\partial y}{\partial u_1} + c\frac{\partial y}{\partial u_2} + d\frac{\partial^2 y}{\partial u_1 \partial u_2},$$

$$\frac{\partial^2 y}{\partial u_2^2} = a' y + b'\frac{\partial y}{\partial u_1} + c'\frac{\partial y}{\partial u_2} + d'\frac{\partial^2 y}{\partial u_1 \partial u_2}.$$

It is evident that in any particular problem this form of the system is most useful; however, in the very general questions which we shall treat subsequently, the form (4) in which we first wrote the system seems of great assistance, in spite of the fact that many of the equations may be superfluous.

From what we have already seen, the analogy of homogeneous linear completely integrable systems to ordinary homogeneous linear differential equations in very close. In later sections the similarity will be made to appear even more striking; at present we take occasion to make the following remark. If in system (4) we keep k fixed, we have a system of n differential equations which may be regarded as a system of ordinary differential equations in which the independent variable is u_k and the dependent variables are n in number, viz., $y, y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$. *The completely integrable system* (4) *may there-*

fore be regarded in p ways as a system of n ordinary differential equations in n dependent variables, containing p − 1 parameters.

4. The integrability conditions and wronskian of a completely integrable system

In the last section, the completely integrable system (4) was set up on the supposition that a fundamental system of solutions was known. Naturally, if a system of this form, which we shall write

$$\frac{\partial y^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} a_i^{(j,k)} y^{(i)} \quad (j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p), \tag{7}$$

is to be completely integrable and have precisely n fundamental solutions, the coefficients cannot be arbitrary. We shall in the present section investigate certain relations which exist among the coefficients of system (7), under the supposition that the system has a fundamental system of n solutions $y_1, y_2, \cdots, y_n$ in the region A.

Let us call the quantities $y, y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ which occur in the right-hand members of equations (7), the *primary derivatives* of the function y.

The determinant W given by equation (3) is formed of the primary derivatives of a fundamental system of solutions of the differential equations. We shall call this determinant the *wronskian* of the system of solutions. In setting up the completely integrable system (4) we assumed that the wronskian of the fundamental system of solutions vanished nowhere in A; this is equivalent to the assumption that *if for every solution of the system of differential equations a relation exists among the primary derivatives, of the form*

$$\alpha_0 y + \alpha_1 y^{(1)} + \alpha_2 y^{(2)} + \cdots + \alpha_{n-1} y^{(n-1)} \equiv 0, \tag{8}$$

then all of the functions $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ must be identically zero.

We shall from now on assume that in the region A the coefficients $a_i^{(j,k)}$ of system (7) are bounded and possess all partial derivatives of the first order. Then since by supposition there exists a function y not identically zero which satisfies equations (7), we may differentiate these equations and obtain in two ways an expression for $\partial^2 y^{(j)}/\partial u_k\, \partial u_l$ $(k \neq l)$. In fact,

$$\frac{\partial^2 y^{(j)}}{\partial u_k\, \partial u_l} = \sum_{i=0}^{n-1} \left(\frac{\partial a_i^{(j,k)}}{\partial u_l} y^{(i)} + a_i^{(j,k)} \frac{\partial y^{(i)}}{\partial u_l} \right),$$

$$\frac{\partial^2 y^{(j)}}{\partial u_l\, \partial u_k} = \sum_{i=0}^{n-1} \left(\frac{\partial a_i^{(j,l)}}{\partial u_k} y^{(i)} + a_i^{(j,l)} \frac{\partial y^{(i)}}{\partial u_k} \right).$$

In the right-hand members of each of these, the derivatives of $y^{(i)}$ may be replaced by their expressions in terms of $y, y^{(1)}, \cdots, y^{(n-1)}$ taken from system (7). In this way two different expressions of the form (8) are obtained for a derivative $\partial^2 y^{(j)}/\partial u_k \partial u_l$ $(k \neq l)$. But these two expressions are to be identically equal, so that by our assumption the coefficients of $y^{(i)}$ in the two expressions must be identically equal. We may therefore state the theorem:

THEOREM VI. *A necessary condition that system* (7) *have a fundamental system of n solutions is that the coefficients satisfy identically the relations*

$$\frac{\partial a_\nu^{(j,k)}}{\partial u_l} + \sum_{i=0}^{n-1} a_i^{(j,k)} a_\nu^{(i,l)} = \frac{\partial a_\nu^{(j,l)}}{\partial u_k} + \sum_{i=0}^{n-1} a_i^{(j,l)} a_\nu^{(i,k)} \tag{9}$$

$$(\nu, j = 0, 1, \cdots, n-1;\ k, l = 1, 2, \cdots, p).$$

We shall speak of equations (9) as the *conditions of complete integrability,* or as the *integrability conditions,* of system (7).

From the set of integrability conditions (9) let us pick out n equations, by keeping k and l fixed and choosing $\nu = j = 0, 1, \cdots, n-1$. The equations are

$$\frac{\partial a_j^{(j,k)}}{\partial u_l} + \sum_{i=0}^{n-1} a_i^{(j,k)} a_j^{(i,l)} = \frac{\partial a_j^{(j,l)}}{\partial u_k} + \sum_{i=0}^{n-1} a_i^{(j,l)} a_j^{(i,k)}. \quad (j = 0, 1, \cdots, n-1).$$

Adding them, we find without difficulty that all terms drop out except the first derivatives, and in fact

$$\sum_{j=0}^{n-1} \frac{\partial a_j^{(j,k)}}{\partial u_l} = \sum_{j=0}^{n-1} \frac{\partial a_j^{(j,l)}}{\partial u_k} \qquad (k, l = 1, 2, \cdots, p),$$

that is,

$$\frac{\partial}{\partial u_l} \sum_{j=0}^{n-1} a_j^{(j,k)} = \frac{\partial}{\partial u_k} \sum_{j=0}^{n-1} a_j^{(j,l)}. \tag{10}$$

Consequently, the n quantities $\sum_{j=0}^{n-1} a_j^{(j,k)}$ are the first derivatives of some function $f(u_1, u_2, \cdots, u_p)$, and we may write

$$\frac{\partial f}{\partial u_k} = \sum_{j=0}^{n-1} a_j^{(j,k)} \qquad (k = 1, 2, \cdots, p). \tag{11}$$

We shall presently find relations (10) and (11) very useful.

Let us now form the wronskian (3) of any fundamental system of n solutions $y_1, y_2, \cdots, y_n$ of system (7):

$$W = \begin{vmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{vmatrix}$$

We differentiate W with respect to u_k; if the result be written as the sum of the n determinants formed from W by replacing a row thereof by the derivatives of that row with respect to u_k, and if use be then made of equations (7) to replace any derivative $\partial y_i^{(j)}/\partial u_k$ by an expression linear in y_i, $y_i^{(1)}, \cdots, y_i^{(n-1)}$, it will be found without difficulty that

$$\frac{\partial W}{\partial u_k} W = \sum_{j=0}^{n-1} a_j^{(j, k)} = \frac{\partial f}{\partial u_k} W \qquad (k = 1, 2, \cdots, p).$$

Consequently,

$$W = const. \; e^f .$$

We may therefore state the theorem, which is a direct generalization of the familiar theorem of Abel in the theory of an ordinary homogeneous linear differential equation of the nth order:

THEOREM VII. *The wronskian W of a fundamental system of n solutions of the completely integrable system* (7) *may be determined by a quadrature from the coefficients of the system, and is given by the expression*

$$W = const. \; e^f , \tag{12}$$

where f may be found by a quadrature in virtue of equations (10) *or* (11).

As a corollary we have the theorem:

THEOREM VIII. *If the wronskian vanish at any point of the region A, it vanishes identically in A.**

We remark in passing that equations (10) and (11) might have been deduced directly from a consideration of the wronskian. For, equations (10) may be deduced by differentiation of the relations

$$\frac{\partial W}{\partial u_k} = W \sum_{j=0}^{n-1} a_j^{(j, k)} \qquad (k = 1, 2, \cdots, p). \tag{13}$$

But the relations (13) are of interest in themselves, because they show that the function $\sum_{j=0}^{n-1} a_j^{(j, k)}$ is the derivative, with respect to u_k, of a function f, for $k = 1, 2, \cdots, p$, and this *without necessitating the assumption of the existence of any of the first derivatives of the coefficients $a_i^{(j, k)}$ of system* (7).

5. THE NORMAL FORM OF THE COMPLETELY INTEGRABLE SYSTEM

A necessary condition that a system of partial differential equations of the form (7) have a solution not identically zero is that the coefficients of the system satisfy the integrability conditions (9). Whether the condition is also sufficient remains to be seen; as a matter of fact, however, an example will show that a system of form (7), whose coefficients satisfy the integrability conditions, need not have a solution which is not identically zero. For, if we select as primary derivatives y and $\partial^2 y/\partial u^2$, and suppose the independent variables to be u and v, we may write down the system of form (7):

* We denote by A the region in which all of the solutions $y_1, y_2, \cdots, y_n$ exist.

$$\frac{\partial y}{\partial u} = ay + b\frac{\partial^2 y}{\partial u^2}, \qquad \frac{\partial y}{\partial v} = a' y + b'\frac{\partial^2 y}{\partial u^2},$$

$$\frac{\partial^3 y}{\partial u^3} = a'' y + b''\frac{\partial^2 y}{\partial u^2}, \qquad \frac{\partial^3 y}{\partial u^2 \partial v} = a''' y + b'''\frac{\partial^2 y}{\partial u^2},$$

which has no solution different from zero even if the integrability conditions of form (9) are satisfied for the system, unless of course other relations exist among the coefficients. In fact, differentiation of the first equation of the system twice with respect to u will lead to two new equations which are in general incompatible with the first and third equations of the system. In our example this results from the peculiar choice of primary derivatives, a gap occurring between y and $\partial^2 y/\partial u^2$.

We shall presently show that it is always possible, for a system of form (7) which is known to have n fundamental solutions, to make a particular choice of primary derivatives which removes the above difficulty, although at a slight sacrifice in generality. Let us first define what we shall call a *chain* of derivatives and a *normal set* of derivatives. A *chain* of derivatives of a function y is a set of derivatives $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, such that $y^{(i)}$ is of the ith order and may be obtained by a differentiation from $y^{(i-1)}$. Thus, denoting partial differentiation by subscripts, we see that the quantities y, y_u, y_{uv}, y_{uuv} form a chain, while y, y_u, y_{uv}, y_{uuu} do not because y_{uuu} cannot be formed from y_{uv} by differentiation.

A *normal set* of derivatives is a set such that each element of the set belongs to at least one chain which can be formed from elements of the set. The normal set need not consist of a single chain, and a given member of the set may belong to more than one chain.

We shall now show that if system (7) has exactly n fundamental solutions $y_1, y_2, \cdots, y_n$, the primary derivatives may under certain conditions be so chosen as to form a normal set. However, in order not to interrupt our reasoning later, we shall first prove a lemma which is to be used in the course of the discussion.

LEMMA. *Suppose that system* (7) *has n solutions* $y_1, y_2, \cdots, y_n$ *whose wronskian W vanishes nowhere in the region A. Then* $y_1, y_2, \cdots, y_n$ *cannot all be solutions of a system R of form* (7) *having less than n primary derivatives, these primary derivatives being selected from among those of system* (7).

Let $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$ $(r < n-1)$ be the primary derivatives of the system R. Then the lemma follows at once if in the matrix

$$M \equiv M_r(y_1, y_2, \cdots, y_n)$$

at least one of the $(r+1)$-rowed determinants vanishes at no point of A, for then this determinant plays the rôle of a wronskian for system R. Suppose, however, that each of the $(r+1)$-rowed determinants vanishes somewhere in A. They can not all be identically zero in A, because then the wronskian of $y_1, y_2, \cdots, y_n$ would be zero also, contrary to hypothesis. Consequently one of these determinants, say that formed from the first $r+1$ columns of M, is different from zero at some point of A, viz., $(a_1, \cdots, a_p)$, and therefore, since it is continuous, it differs from zero in a p-dimensional region A' containing this point. Then in this region A', every solution of system R is linearly dependent on $y_1, y_2, \cdots, y_{r+1}$, so that in A' every solution of (7), being also a solution of R, will be linearly dependent on $y_1, y_2, \cdots, y_{r+1}$. Then in particular we shall have in A'

$$y_{r+2} = c_1 y_1 + c_2 y_2 + \cdots + c_{r+1} y_{r+1},$$

the c's being constants not all zero. Therefore the function

$$y = y_{r+2} - c_1 y_1 - c_2 y_2 - \cdots - c_{r+1} y_{r+1}$$

will vanish identically in A', and so will its derivatives $y^{(1)}, y^{(2)}, \cdots, y^{(n)}$. But it is easily seen that if system (7) has n solutions $y_1, y_2, \cdots, y_n$ for which the wronskian vanishes nowhere in A, then any solution all of whose primary derivatives vanish at some point of A will vanish identically in A. Consequently the function y considered above vanishes identically not only in A', but throughout A. Therefore we conclude that system (7) would have only $r+1$ linearly independent solutions, which contradicts the hypothesis.

We now proceed to show how system (7) may be reduced to a similar system for which the primary derivatives form a normal set, under the hypothesis that system (7) has exactly n fundamental solutions $y_1, y_2, \cdots, y_n$ whose wronskian W does not vanish at any point of a region A of p dimensions. We assume, then, that the n primary derivatives of system (7) do not form a normal set. Then we may select from among these primary derivatives a normal set which consists of at least one element, namely y; let this normal set be $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, $(r < n-1)$. Suppose, furthermore, that it is the *largest* normal set which may be formed from among the primary derivatives. Then of the first derivatives of $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$ none is a primary derivative of system (7), for otherwise our normal set could be enlarged.* Now, all of the first derivatives just mentioned are expressible linearly in terms of the primary derivatives of system (7), the coefficients in the expressions being continuous in A. This gives us $p(r+1)$ equations, in the right-hand members of which appear some or all of the quantities $y, y^{(1)}, y^{(2)}, \cdots,$

* Lest a misunderstanding occur here, we point out that if $y^{(l)}$ and $y^{(m)}$ are primary derivatives, and if $\partial y^{(l)}/\partial u_i = y^{(m)}$, then this equation is one of the equations of system (7); also, if $y^{(l)}$ be one of the normal set $y, y^{(1)}, \cdots, y^{(r)}$ under consideration, then $y^{(m)}$ belongs to the said normal set also.

$y^{(r)}$. But in the right-hand member of at least one of these equations must appear some one of the primary derivatives, say $y^{(s)}$, which is not one of our normal set, for otherwise the $p(r+1)$ equations would constitute a system of form (7) whose primary derivatives are $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, i. e., the said $p(r+1)$ equations would constitute a system R such as is considered in the lemma proved above, and the original system (7) could then have no more than $r+1$ linearly independent solutions, contrary to hypothesis. Consequently, of the first derivatives of $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, at least one has a linear expression, in terms of the primary derivatives, which contains a primary derivative $y^{(s)}$ which is not a member of the normal set $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$. In other words, the coefficient of $y^{(s)}$ in this expression is not identically zero; since this coefficient is continuous, there must exist a p-dimensional sub-region of A, say A_1, at no point of which it vanishes. Consequently, in the region A_1 we may express the primary derivative $y^{(s)}$ linearly in terms of all the other primary derivatives, and the additional quantity which occurs in the left-hand member of the equation from which we have just found the expression for $y^{(s)}$. This additional quantity, we recall, is one of the first derivatives of one of the elements of the normal set $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$, and is not itself a primary derivative. Let us denote it by $y^{(t)}$. Then throughout (7) we may replace $y^{(s)}$ everywhere by its equivalent expression, and thus obtain a new system of differential equations, of form (7), in which the primary derivatives are the same except that $y^{(t)}$ has taken the place of $y^{(s)}$. This new set therefore contains in it a normal set composed of the previous normal set $y, y^{(1)}, y^{(2)}, \cdots, y^{(r)}$ and the additional derivative $y^{(t)}$. The new wronskian of the system of differential equations vanishes nowhere in the p-dimensional region A_1. But although the coefficients in system (7) are supposed to have all of their first derivatives, it is logically possible that all of the first derivatives do not exist for the coefficients of the new system. We shall suppose that these derivatives do exist.* Then on the new system of differential equations we may proceed as before, enlarging step by step the number of primary derivatives which form a normal set until finally a system is obtained for which all of the n primary derivatives form a normal set. In this reduction we have supposed of course that certain derivatives exist for the coefficients in system (7). In any event, it is clear that such derivatives need not exist of order higher than $n-r-1$, where r is the number which appears in the above discussion.

We shall henceforth suppose that the primary derivatives in system (7)

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* The writer does not know whether they exist as a consequence of the previous hypotheses on system (7) or not. Since the form to which we shall ultimately reduce our system seems essential for the proof of the existence theorem, it would be of interest to have this point settled.

form a normal set, and shall then speak of system (7) as being in the *normal form.* The chief importance of the normal form lies in the fact that system (7) in that form may be regarded as a system of np partial differential equations of the first order in the p independent variables $u_1, u_2, \cdots, u_p$, and the n dependent variables $y, y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$, since among equations (7) occur a certain number which define each primary derivative as the first derivative of another primary derivative belonging to the same chain.*

6. The canonical form of a completely integrable system

It is well known that by a transformation of the dependent variable of the form $y = \lambda(x)\bar{y}$ any ordinary homogeneous linear differential equation of the nth order,

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

whose coefficients are continuous functions of x, may be transformed into one of the same form in which the coefficient p_1 is zero. We shall now show that a similar thing may be done for certain completely integrable systems.

We introduce a new dependent variable, $\bar{y}$, connected with the old by the equation

$$y = \lambda\bar{y},$$

where λ is a function of $u_1, u_2, \cdots, u_p$. Any derivative of y, say $y^{(j)}$, is connected with derivatives of the new dependent variable, $\bar{y}$, by an equation of the form

$$y^{(j)} = \lambda\bar{y}^{(j)} + \cdots,$$

where the part omitted in the right-hand member is linear in derivatives of $\bar{y}$, and contains all those derivatives of $\bar{y}$ from which $\bar{y}^{(j)}$ may be obtained by differentiation.

Let us now carry out this transformation on a particular kind of completely integrable system, namely, one in which the primary derivatives not only form a normal set, but have the additional property that, if a derivative $y^{(j)}$ belong to the set, then *every* derivative of lower order from which $y^{(j)}$ is obtainable by successive differentiation is likewise a primary derivative. It is not difficult to see that in this case, if y be replaced everywhere by $\lambda\bar{y}$, the system of differential equations is changed into a new system, with the dependent variable $\bar{y}$, which new system may be thrown into exactly the same form as the original system. Let us write the new system

$$\frac{\partial \bar{y}^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} \bar{a}_i^{(j,k)} \bar{y}^{(i)} \tag{14}$$
$$(j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p),$$

* See footnote on page 503.

where the coefficients $\bar{a}_i^{(j,k)}$ are functions of the old coefficients $a_i^{(j,k)}$ (of system (7)) and also of λ and its successive derivatives. Of course the actual expressions depend upon the form of each individual primary derivative, which form we have not specified.

We may, however, obtain explicit results with regard to the transformation of the wronskian and its first derivatives. Let us denote by $\overline{W}$ the wronskian for the transformed system (14). We recall that the set of primary derivatives has the property that if $y^{(j)}$ belong to the set all the derivatives of y from which $y^{(j)}$ may be obtained by differentiation also belong to the set. Consequently the wronskian $\overline{W}$ for the new system is easily seen to be related to the wronskian for the old system by the equation

$$W = \lambda^n \overline{W}. \tag{15}$$

Differentiating this equation with respect to u_k, we have

$$\frac{\partial W}{\partial u_k} = \lambda^n \frac{\partial \overline{W}}{\partial u_k} + n\lambda^{n-1} \overline{W} \frac{\partial \lambda}{\partial u_k},$$

or, using equations (13) and (11),

$$\frac{\partial f}{\partial u_k} W = \lambda^n \overline{W} \frac{\partial \bar{f}}{\partial u_k} + n\lambda^{n-1} \overline{W} \frac{\partial \lambda}{\partial u_k},$$

where $\bar{f}$ denotes the function which takes the place of f for the new system of differential equations (14). Substituting for $\overline{W}$ its value as obtained from (15), and recalling that W is nowhere zero in the region A, we find that

$$\lambda \frac{\partial \bar{f}}{\partial u_k} = \lambda \frac{\partial f}{\partial u_k} - n \frac{\partial \lambda}{\partial u_k} \qquad (k = 1, 2, \cdots, p).$$

This result enables us to state the theorem:

THEOREM IX. *Suppose a system of partial differential equations of the form* (7) *has the following properties*:

1°. *It is completely integrable, with n fundamental solutions whose wronskian* (3) *vanishes at no point of A.*

2°. *The set of primary derivatives is such that, if $y^{(j)}$ be any one of the set, then all the derivatives of lower order from which $y^{(j)}$ may be obtained by differentiation also belong to the set.*

3°. *All of the primary derivatives exist for each of the np coefficients $a_j^{(j,k)}$* $(j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p)$.

Then the system of differential equations (7) *may be transformed into another uniquely determined completely integrable system* (14) *for which all of the quantities*

$$\frac{\partial \bar{f}}{\partial u_k} \equiv \sum_{j=0}^{n-1} \bar{a}_j^{(j,k)} \qquad (k = 1, 2, \cdots, p),$$

are zero, by the transformation of the dependent variable $y = \lambda \bar{y}$, *where*

$$\lambda \frac{\partial f}{\partial u_k} - n \frac{\partial \lambda}{\partial u_k} = 0, \qquad \lambda = \text{const. } e^{f/n}. \tag{16}$$

The system (14) thus obtained will be unique, and its coefficients $\bar{a}_i^{(j,k)}$ will be explicit expressions in the coefficients of system (7) and their derivatives. Let us call this uniquely determined system the *canonical form* of system (7).

It should be noted that part 3° of the hypothesis is necessary in order that $\lambda \bar{y}$ may be substituted for y throughout equation (7).

Let us return to the general transformation $y = \lambda \bar{y}$, which converts system (7) into system (14); we now suppose that system (14) is not in the canonical form. The coefficients of system (14) are expressible explicitly in terms of the coefficients of (7) and their derivatives, and λ and its derivatives. Consider any function $\bar{F}$ of the coefficients of (14) and their derivatives. Let F be the same function of the corresponding coefficients of (7) and their derivatives. The function $\bar{F}$ is equal identically to a function of λ and the coefficients of (7), and the derivatives of these quantities. If in particular this latter function is related to $\bar{F}$ by the identity

$$F = \lambda^r \bar{F},$$

where r may or may not be zero, we call the function F a *seminvariant* of system (7).

The following theorem is a consequence of the fact that system (7) and the transformed system (14) have exactly the same canonical form:

THEOREM X. *The coefficients of the canonical form are seminvariants of system* (7). *All other seminvariants are functions of these and of their derivatives, if such derivatives exist.*

The reduction of a completely integrable system to the canonical form is seen by the above discussion to be a purely mechanical process, performable in any case coming under the hypotheses of Theorem IX. This fact is of practical importance in projective differential geometry, if the general method of Wilczynski be followed. We shall return to this question in the next section, applying the method to a particular geometric problem.

In § 8, we shall give an existence theorem for systems of form (7) in which the primary derivatives constitute a normal set. It should be noted that in the hypothesis 2° of Theorem IX, however, the set of primary derivatives is still further restricted; as a matter of fact, we may show by an example that this restriction is in general necessary, so that not every completely integrable system may be reduced to a unique canonical form. Denoting by subscripts partial differentiation with respect to the two independent variables u and v, let the primary derivatives be y, y_u, y_{uv}. These form a normal set. If y_1, y_2, y_3 be functions of u, v for which in the region A the wronskian

$$W = |y, y_u, y_{uv}|$$

is nowhere zero, we may set up the completely integrable system having y_1, y_2, y_3 as a fundamental set of solutions:

$$\begin{aligned} y_v &= ay + by_u + cy_{uv}, \\ y_{uu} &= a' y + b' y_u + c' y_{uv}, \\ y_{uuv} &= a'' y + b'' y_u + c'' y_{uv}, \\ y_{uvv} &= a''' y + b''' y_u + c''' y_{uv}, \end{aligned}$$

where in general $c \neq 0$. Making the transformation $y = \lambda\bar{y}$, we find that W is transformed into

$$W = |\lambda\bar{y}, \lambda\bar{y}_u + \lambda_u \bar{y}, \lambda\bar{y}_{uv} + \lambda_v \bar{y}_u + \lambda_u \bar{y}_v + \lambda_{uv} \bar{y}| = (\lambda^3 + c\lambda^2 \lambda_u) \overline{W},$$

in which use has been made of the first of the system of differential equations. Now, if $c \neq 0$—a condition which may be realized at pleasure—this relation between W and $\overline{W}$ is not of the form (15), so that the differential equation (16) which determines λ is replaced by one of the second order. Therefore, although the system may be reduced to the canonical form, this form will not be unique.

Of course, instead of making the said hypothesis 2° of Theorem IX, we might demand merely that the wronskian and its transform be related by equation (15). Our example would then be included in this case, if c were identically zero in A.

7. Geometric application; curvilinear coördinates*

Completely integrable systems of the kind we have been considering have attained increased importance of late owing to their systematic use in certain fields of projective differential geometry. Since any solution of the system is linearly dependent on any fundamental set of solutions, it follows that a geometric configuration defined by a fundamental set of solutions is a projective transformation of the configuration defined by any other fundamental set of solutions. Consequently, any geometric property expressed in terms of the completely integrable system, being common to all the projective transformations of the configuration defined by a fundamental set of solutions of the system, is a projective property. Wilczynski has developed a general method for dealing with certain questions from this point of view, and it

* This section is independent of § 8.

is in connection with this method that the results of § 6 find an important application.

Let us consider in particular the general subject of curvilinear coördinates in space of n dimensions.* If $y_1, y_2, \cdots, y_{n+1}$ be interpreted as homogeneous coördinates of a point in this space, the equations

$$y_i = y_i(u_1, u_2, \cdots, u_n) \qquad (i = 1, 2, \cdots, n+1) \tag{17}$$

will define a system of curvilinear coördinates, provided the ratios of the y's do not all reduce to functions of fewer than n of the independent variables $u_1, u_2, \cdots, u_n$. We shall suppose that all derivatives of any order exist for the functions y_i. Since at least one of these functions, say y_{n+1}, is different from zero at some point, it will be different from zero in an n-dimensional region about this point. We wish, then, the analytic condition that the n ratios $\eta_i \equiv y_i/y_{n+1}$, $(i = 1, 2, \cdots, n)$ actually define ∞^n points. This condition is the non-vanishing of the jacobian of the functions η_i, and this may be reduced to the non-vanishing of the determinant

$$W = \begin{vmatrix} y_1, & y_2, & \cdots, & y_{n+1} \\ y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_{n+1}^{(1)} \\ y_1^{(2)}, & y_2^{(2)}, & \cdots, & y_{n+1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n)}, & y_2^{(n)}, & \cdots, & y_{n+1}^{(n)} \end{vmatrix}, \tag{18}$$

where $y_1^{(j)}$, $y_2^{(j)}$, etc., denote $\partial y_1/\partial u_j$, $\partial y_2/\partial u_j$, etc. We may therefore set up a completely integrable system of form (7) of which the $n+1$ functions y_i constitute a fundamental set of solutions, and in which the primary derivatives are $y, y^{(1)}, y^{(2)}, \cdots, y^{(n)}$. This system is

$$\frac{\partial y}{\partial u_k} = y^{(k)}, \qquad \frac{\partial y^{(j)}}{\partial u_k} = \sum_{i=0}^{n} a_i^{(j,k)} y^{(i)} \qquad (j, k = 1, 2, \cdots, n). \tag{19}$$

Let us differentiate W with respect to u_k; then on using equations (19) we find without difficulty that

$$\frac{\partial W}{\partial u_k} = (a_1^{(1,k)} + a_2^{(2,k)} + \cdots + a_n^{(n,k)})\, W.$$

We may therefore put

$$\sum_{j=1}^{n} a_j^{(j,k)} \equiv \frac{\partial f}{\partial u_k} \qquad (k = 1, 2, \cdots, n).$$

* The case $n = 2$, i. e., nets of plane curves, has been studied by Wilczynski, *One-parameter families and nets of plane curves*, these Transactions, vol. 12 (1911), pp. 473–510. The present writer considered the case $n = 3$ in his Columbia dissertation, *Projective differential geometry of triple systems of surfaces*, Lancaster, Pa., Press of The New Era Printing Company, 1913.

Now, there is an arbitrary feature introduced in equations (17) because of the homogeneity of the coördinates. If λ be any function of $u_1, u_2, \cdots, u_n$, then the points defined by the coördinates $\bar{y}_i$ $(i = 1, 2, \cdots, n+1)$ will be the same as those defined by equations (17), if $y_i = \lambda\bar{y}_i$. Consequently, if a projective property of the configuration is to be expressed in terms of the coefficients of the completely integrable system (19), it must be in terms of what we called in § 6 the *seminvariants* of the system, viz., those combinations of the coefficients of equations (19) and their derivatives which remain essentially unchanged when the system is subjected to an arbitrary transformation of the form

$$y = \lambda\bar{y}.$$

Let us carry out this transformation of the dependent variable, thus converting system (19) into one of the same form,

$$\frac{\partial\bar{y}}{\partial u_k} = \bar{y}^{(k)}, \qquad \frac{\partial\bar{y}^{(j)}}{\partial u_k} = \sum_{i=0}^{n} \bar{a}_i^{(j,k)}\,\bar{y}^{(i)} \quad (j, k = 1, 2, \cdots, n). \tag{20}$$

The coefficients of this system are functions of the coefficients of (19) and their derivatives, and also of λ and its derivatives. In particular,

$$\bar{a}_j^{(j,k)} = a_j^{(j,k)} - \lambda^{(k)}/\lambda \quad (j \neq k), \qquad \bar{a}_j^{(j,j)} = a_j^{(j,j)} - 2\lambda^{(k)}/\lambda$$
$$(j, k = 1, 2, \cdots, n),$$

so that, on denoting by $\partial\bar{f}/\partial u_k$ the quantity $\sum_{j=1}^{n} \bar{a}_j^{(j,k)}$, we find directly that

$$\lambda\frac{\partial\bar{f}}{\partial u_k} = \lambda\frac{\partial f}{\partial u_k} - (n+1)\frac{\partial\lambda}{\partial u_k} \qquad (k = 1, 2, \cdots, n).$$

This also follows from the general results of § 6. Consequently, the quantities $\bar{f}^{(k)} \equiv \partial\bar{f}/\partial u_k$ will all be zero if λ be chosen as a solution of

$$\lambda f^{(k)} - (n+1)\lambda^{(k)} = 0 \qquad (k = 1, 2, \cdots, n),$$

which gives

$$\lambda = \text{const. } e^{f/(n+1)}.$$

With this choice of λ the coefficients of the new system (20) have the values

$$\begin{aligned}
&\bar{a}_0^{(j,j)} = a_0^{(j,j)} + \frac{1}{n+1}\left(\sum_{i=1}^{n} a_i^{(j,j)} f^{(i)} - \frac{\partial f^{(j)}}{\partial u_j} - \frac{1}{n+1} f^{(j)} f^{(j)}\right),\\
&\bar{a}_i^{(j,j)} = a_i^{(j,j)} \ (i \neq j), \qquad \bar{a}_j^{(j,j)} = a_j^{(j,j)} - 2f^{(j)},\\
&\bar{a}_0^{(j,k)} = a_0^{(j,k)} + \frac{1}{n+1}\left(\sum_{i=1}^{n} a_i^{(j,k)} f^{(i)} - \frac{\partial f^{(j)}}{\partial u_k} - \frac{1}{n+1} f^{(j)} f^{(k)}\right),\\
&\bar{a}_i^{(j,k)} = a_i^{(j,k)} \ (i \neq j, k), \qquad \bar{a}_j^{(j,k)} = a_j^{(j,k)} - f^{(k)},\\
&\bar{a}_k^{(j,k)} = a_k^{(j,k)} - f^{(j)} \qquad (j \neq k;\ i, j, k = 1, 2, \cdots, n),
\end{aligned} \tag{21}$$

where throughout single superscripts denote differentiation. The quantities (21) are the fundamental seminvariants of system (19), and any seminvariant of the system is a function of these and their derivatives.

There still remains, however, an arbitrary feature in equations (17). This is in the choice of parameters. Since the system of curvilinear coördinates remains unchanged under the transformation

$$\bar{u}_k = U_k(u_k), \qquad (k = 1, 2, \cdots, n), \tag{22}$$

it follows that projective properties of the configuration are expressible in terms of those seminvariants which remain essentially invariant when system (19) is subjected to the transformation (22). It is possible to calculate explicitly, and without any difficulty, these *invariants*, but our object here is not to give a systematic theory of curvilinear coördinates.

The fact that even in this very general problem the invariants may be calculated explicitly, makes it highly probable that the analytic methods of Wilczynski are practicable in handling problems in the projective differential geometry, not only of configurations in spaces of a few dimensions, but also in space of n dimensions. The only methods which have hitherto been used systematically in studying configurations in hyperspace are those of Segre.* The ideal method seems to the present writer to be one in which all the projective differential properties are expressible by means of a unique analytic apparatus, and this is to be found in the completely integrable system of which a fundamental system of solutions defines the particular geometric configuration under discussion.

8. The existence theorem for completely integrable systems. The linear dependence of solutions

The theorems of previous sections concerning the system of partial differential equations (7) presuppose the existence of a fundamental system of n solutions. The question of this existence presents no essentially new difficulty, for the remark made at the end of § 5 renders immediately applicable to system (7), which is supposed to be in the normal form, a well-known existence theorem concerning systems of total differential equations.†

* For references to some of this work, see E. Bompiani, *Recenti progressi nella geometria proiettiva differenziale degli iperspazi*, Proceedings of the fifth international congress of mathematicians, Cambridge, 1912, vol. II, pp. 22–27.

In this connection should be mentioned also Prof. Wilczynski's Cleveland address, *Some general aspects of modern geometry*, Bulletin of the American Mathematical Society, vol. 19 (1913–14), pp. 331–342.

† Cf., for example, Ch.-J. de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 2me éd., vol. 2, chap. VII, § 6. The theorem is essentially that of Mayer, given in the memoir cited in § 3.

A simultaneous system of n total differential equations in n dependent variables x_h $(h = 1, 2, \cdots, n)$ and m independent variables y_i $(i = 1, 2, \cdots, m)$ is of the form

$$dx_h = \sum_{i=1}^{n} a_h^{(i)} dy_i \quad (h = 1, 2, \cdots, n;\ i = 1, 2, \cdots, m), \tag{23}$$

the a's being continuous functions of the x's and y's in a region A. Let us write

$$\frac{\delta}{\delta y_k} = \frac{\partial}{\partial y_k} + \sum_{h=1}^{n} a_h^{(k)} \frac{\partial}{\partial x_k}.$$

Then we shall say that system (23) is *completely integrable* if its coefficients satisfy the identities

$$\frac{\delta a_h^{(i)}}{\delta y_k} = \frac{\delta a_h^{(k)}}{\delta y_i} \quad (i, k = 1, 2, \cdots, m;\ h = 1, 2, \cdots, n). \tag{24}$$

We shall now state the existence theorem for this system.

THEOREM. *Suppose that the identities* (24) *subsist for system* (23) *throughout the region* A, *and let* $y_1^{(0)}, \cdots, y_m^{(0)};\ x_1^{(0)}, \cdots, x_n^{(0)}$ *be a set of values of the independent and dependent variables in the interior of this region. Then there exists one and only one set of n functions* $x_1, \cdots, x_n$ *of the m independent variables* $y_1, \cdots, y_m$, *which satisfy the system of differential equations and which respectively take on the values* $x_1^{(0)}, \cdots, x_n^{(0)}$ *when* $y_1 = y_1^{(0)}, \cdots, y_m = y_m^{(0)}$.

Consider now system (7), viz.,

$$\frac{\partial y^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} a_i^{(j,k)} y^{(i)} \quad (j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p),$$

in which we suppose that there are exactly n primary derivatives, and that they form a normal set. Let us regard $y, y^{(1)}, \cdots, y^{(n-1)}$ as dependent variables; then the system becomes a very particular case of the system (23) which we have just been considering. The integrability conditions (24) take precisely the form of equations (9). Moreover, from the remark made at the end of § 5, system (7) when regarded as a system of partial differential equations of the first order with the dependent variables $y, y^{(1)}, \cdots, y^{(n-1)}$ is exactly equivalent to system (7) with its original significance, viz., that in which $y, y^{(1)}, \cdots, y^{(n-1)}$ are primary derivatives of the function y, each superscript denoting a certain particular, though unspecified, partial derivative of y. The exact equivalence of the two points of view follows from the fact that the primary derivatives form a normal set. In other words, if system (7) be regarded as a system of partial differential equations of the first order in n dependent variables, then a set of solutions $y, y^{(1)}, \cdots, y^{(n-1)}$ will have the property that $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ are exactly those derivatives of y which the original superscript notation signified; and all this without requiring the addition of new equations to system (7).

The force of these remarks, especially the last, may be seen by means of the completely integrable system in two independent variables u and v, with primary derivatives y, y_u, y_v:

$$\text{(a)}\qquad \begin{aligned} y_{uu} &= ay_u + by_v + cy, \\ y_{uv} &= a' y_u + b' y_v + c' y, \\ y_{vv} &= a'' y_u + b'' y_v + c'' y, \end{aligned}$$

where the subscripts denote differentiation. This system may be replaced by a system with three dependent variables by means of the substitution

$$\text{(b)}\qquad y_u = y^{(1)}, \qquad y_v = y^{(2)},$$

and will then have the form

$$\text{(a')}\qquad \begin{aligned} y_u^{(1)} &= ay^{(1)} + by^{(2)} + cy, \\ y_v^{(1)} &= a' y^{(1)} + b' y^{(2)} + c' y, \\ y_u^{(2)} &= a' y^{(1)} + b' y^{(2)} + c' y, \\ y_v^{(2)} &= a'' y^{(1)} + b'' y^{(2)} + c'' y. \end{aligned}$$

This is not in the form (7), however, because all of the first derivatives of all of the primary derivatives do not appear in the left-hand members of system (a'). But equations (a') and (b) together constitute a system of form (7). System (a') alone is not equivalent to system (a), but systems (a') and (b') together are equivalent to system (a).

A better illustration is afforded by the system of form (7) introduced at the beginning of § 5, viz., one in which the primary derivatives are y and y_{uu}, and which we there saw had no solution different from zero. If we put

$$y_{uu} = y^{(1)},$$

then the system in form (7) is

$$\begin{aligned} y_u &= ay + by^{(1)}, &\qquad y_v &= a' y + b' y^{(1)}, \\ y_u^{(1)} &= a'' y + b'' y^{(1)}, &\qquad y_v^{(1)} &= a''' y + b''' y^{(1)}. \end{aligned}$$

These four equations, regarded as equations of the first order in the dependent variables $y, y^{(1)}$, will have sets of solutions $y, y^{(1)}$ different from zero if the integrability conditions (9) are satisfied for the system. But the function $y^{(1)}$ will not in general be the proper derivative of the function y, viz., y_{uu}. This can not happen in the case of the other example, because equations (b), which form part of the system in the form (7), make it certain that a solution y, $y^{(1)}$, $y^{(2)}$ of the system consisting of equations (a') and (b) will have the property that $y^{(1)}$ and $y^{(2)}$ will be the proper derivatives, y_u and y_v respectively, of y.

Let us, then, regard system (7) as a system of partial differential equations of the first order in the n dependent variables $y, y^{(1)}, \cdots, y^{(n-1)}$. Applying the existence theorem for system (23), of which (7) is a particular case, we may state the following existence theorem:

THEOREM XI. *Suppose that in the system of partial differential equations*

$$(7) \qquad \frac{\partial y^{(j)}}{\partial u_k} = \sum_{i=0}^{n-1} a_i^{(j,k)} y^{(i)} \qquad (j = 0, 1, \cdots, n-1;\ k = 1, 2, \cdots, p)$$

the primary derivatives $y, y^{(1)}, \cdots, y^{(n-1)}$ form a normal set. Suppose further that in the region A the coefficients $a_i^{(j,k)}$, which are functions of the p independent variables $u_1, u_2, \cdots, u_p$, satisfy identically the integrability conditions

$$(9) \qquad \frac{\partial a_\nu^{(j,k)}}{\partial u_l} + \sum_{i=0}^{n-1} a_i^{(j,k)} a_\nu^{(i,l)} = \frac{\partial a_\nu^{(j,l)}}{\partial u_k} + \sum_{i=0}^{n-1} a_i^{(j,l)} a_\nu^{(i,k)} \qquad (\nu, j = 0, 1, \cdots, n-1;\ k, l = 1, 2, \cdots, p).$$

Let $(u_1^{(0)}, u_2^{(0)}, \cdots, u_p^{(0)})$ be any point of A, and $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ be any set of n constants. Then there exists one and only one function y of the variables $u_1, u_2, \cdots, u_p$ which satisfies the system of differential equations, and whose primary derivatives $y, y^{(1)}, \cdots, y^{(n-1)}$ take on respectively the preassigned constant values $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ at the point $(u_1^{(0)}, u_2^{(0)}, \cdots, u_p^{(0)})$.

As in the familiar case of ordinary differential equations, this theorem allows us to infer the existence of n linearly independent solutions whose wronskian vanishes nowhere in A. In fact, we may construct a particular set of solutions having this property, by choosing solutions $Y_1, Y_2, \cdots, Y_n$ such that the primary derivatives of the solution Y_i take on the values

$$Y_i^{(j)} = 0 \ (j \neq i-1), \qquad Y_i^{(j)} = 1 \ (j = i-1)$$

at the point $(u_1^{(0)}, u_2^{(0)}, \cdots, u_p^{(0)})$. At this point the wronskian of $Y_1, Y_2, \cdots, Y_n$ is then

$$\begin{vmatrix} 1, & 0, & \cdots, & 0 \\ 0, & 1, & \cdots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdots, & 1 \end{vmatrix} = 1,$$

so that the wronskian vanishes nowhere in A, by Theorem VIII. Moreover, that solution of the system for which the primary derivatives take on the values $y_0, y_0^{(1)}, \cdots, y_0^{(n-1)}$ at the point $(u_1, u_2, \cdots, u_p)$ is given by

$$y = y_0 Y_1 + y_0^{(1)} Y_2 + y_0^{(2)} Y_3 + \cdots + y_0^{(n-1)} Y_n.$$

If a solution has all of its primary derivatives zero at any point of A, it therefore vanishes identically in A. Let $y_1, y_2, \cdots, y_n$ be any n solutions of the completely integrable system; then no function, other than zero, of the form

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

can vanish together with its derivatives $y^{(1)}, y^{(2)}, \cdots, y^{(n-1)}$ at any point of A, unless the constants c_i are all zero. This set of functions $y_1, y_2, \cdots, y_n$ has therefore the properties of the n functions of Theorem III. We may therefore apply this theorem to prove the following one, which is also a consequence of the existence of n solutions whose wronskian vanishes nowhere in A:

Theorem XII. *If $y_1, y_2, \cdots, y_n$ be any n solutions of the completely integrable system of Theorem XI, then the vanishing, anywhere in A, of the wronskian of these n functions is a sufficient condition for their linear dependence.*

This theorem is included in the following more general one. They are both generalizations of theorems given by Bôcher for ordinary differential equations.*

Theorem XIII. *If $y_1, y_2, \cdots, y_k$ $(k \leqq n)$ be solutions of the completely integrable system of Theorem XI, a sufficient condition that they be linearly dependent in A is the identical vanishing in A of all those k-rowed determinants of the matrix*

$$M \equiv M_{n-1}(y_1, y_2, \cdots, y_k)$$

which contain elements of the first row of the matrix.

The proof of this theorem follows immediately from Theorem IV. The matrix M of Theorem IV is in the present case the matrix formed from the primary derivatives of the functions $y_1, y_2, \cdots, y_k$, while the augmented matrix M' has additional rows consisting of the first derivatives of the quantities appearing in M. But these first derivatives are linear combinations of the primary derivatives, in virtue of the completely integrable system, so that if any ν columns $(\nu \leqq k)$ be selected from M, the vanishing of all the ν-rowed determinants formed from these columns and containing elements of the first row of M carries with it the vanishing of all the ν-rowed determinants formed from the same ν columns of M' and containing elements of the first row of M. Moreover, no solution of the completely integrable system, other than zero, can have all of its primary derivatives vanish simultaneously at any point of A. Consequently all of the hypotheses of Theorem IV are satisfied by our functions $y_1, y_2, \cdots, y_k$, and the present theorem follows at once.

* M. Bôcher, loc. cit., these Transactions, vol. 2 (1901), pp. 139–149, theorem VII.

Enough has been given to show the marked analogy between ordinary homogeneous linear differential equations of the nth order and the completely integrable systems which have come under discussion in this paper. This analogy indicates the direction in which facts concerning ordinary differential equations may be extended to systems of partial differential equations. It is obvious how an adjoint system may be defined; the question naturally presents itself as to whether boundary value problems similar to those for ordinary differential equations may arise in connection with completely integrable systems. Questions concerning analytic functions of several variables suggest themselves also. In particular, can the solutions of a given completely integrable system be characterized by their properties as analytic functions, that is, does there exist a solution of the Riemann problem for such a system? Unfortunately, the theory of functions of several variables has not yet reached a degree of development comparable with that of functions of one variable, so that it is doubtful whether the generalization is feasible at present. However, all of these subjects certainly deserve investigation.

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